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Satake diagrams of affine Kac–Moody algebras

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Abstract

Satake diagrams of affine Kac–Moody algebras (untwisted and twisted) are obtained from their Dynkin diagrams. These diagrams give a classification of restricted root systems associated with these algebras. In the case of simple Lie algebras, these root systems and Satake diagrams correspond to symmetric spaces which have recently found many physical applications in quantum integrable systems, quantum transport problems, random matrix theories etc. We hope these types of root systems may have similar applications in theoretical physics in future and may correspond to symmetric spaces analogue of affine Kac–Moody algebras if they exist.

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1. Introduction

Recent times have witnessed a lot of activity in the study of quantum integrable systems [1] and quantum transport problems [2, 3] which are based on the theory of symmetric spaces [4, 5]. The symmetric spaces through their root systems are related to Calogero–Sutherland models of quantum integrable systems. At the same time, the theory of symmetric space provides a classification of random matrix models [6–10] which are studied in various quantum transport problems. Mainly Wigner–Dyson ensemble, chiral, random matrix transfer ensemble, N S ensemble are linked to one of the eleven classes of symmetric spaces in the Cartan classification scheme. These symmetric spaces along with symmetric spaces related with exceptional Lie algebras can be classified in terms of Satake diagrams which are nothing but modified Dynkin diagrams and correspond to modified root lattices (restricted root systems). At the same time these diagrams also classify all the real forms of complex Lie algebras up to isomorphism. Now it is beyond doubt that Kac–Moody algebras, in particular the affine algebras, have wide physical applications in the context of integrable systems, two dimensional field theories, string theories etc. Keeping this in view, in this paper we have obtained an exhaustive list of Satake diagrams of affine Kac–Moody algebras together with the Dynkin diagrams of the restricted root systems. This will serve two purposes:

first it will be a preliminary stage for one way of classification of restricted root systems associated with affine Kac–Moody algebras. We are not sure but hope that these systems may correspond to symmetric spaces analogue of affine Kac–Moody algebras if they exist. Second, these diagrams can also classify the real forms of affine Kac–Moody algebras much of which has been discussed already by different authors [11–14, 21]. Also, the involutive automorphisms determined from these diagrams [15, 16] have been used successfully for the direct determination of Iwasawa/Langlands decompositions associated with these algebras [17, 18].

The organization of the paper is as follows: in section 2, we give a brief introduction to involutive automorphisms and the real forms of affine Kac–Moody algebras. Here we briefly outline the essential procedure for the construction of Satake diagrams, and we enumerate the exhaustive list of these diagrams together with the Dynkin diagrams of restricted root systems for affine Kac–Moody algebras (both twisted and untwisted). Section 3 contains a few concluding remarks.

2. Involutive automorphisms and real forms of affine Kac–Moody algebra

The involutive automorphisms and the real forms of affine Kac–Moody algebras have been discussed in great details by various authors [11–14, 21–23]. However, for the sake of completeness we enumurate the salient points.

We define a group G acting on a Kac–Moody Lie algebra g via adjoint map. Ad: $G \to \mathrm{Aut}(g)$. It is generated by the subgroups U_α for $\alpha \in \Delta$; Δ being the root system of g and Ad $(U_\alpha) = \exp\{\mathrm{ad}(g_\alpha)\}$, where g_α is the α root space in the Lie algebra g and U_α is a subgroup in the Lie group.

A maximal ad_g-diagonalizable subalgebra of g is called a Cartan subalgebra. Every Cartan subalgebra of g is Ad (G)-conjugate to the standard Cartan subalgebra h. A Borel subalgebra of g is a maximal solvable subalgebra. It is always conjugated by Ad (G) to b^+ or b^-, where

$$b^+ = h \bigoplus_{\alpha>0} g_\alpha \tag{2.1}$$

$$b^- = h \bigoplus_{\alpha<0} g_\alpha. \tag{2.2}$$

But b^+ and b^- are not conjugated under Ad (G), so there are exactly two conjugacy classes of Borel subalgebra: the positive and negative subalgebras.

If $\mathrm{Aut}(g)$ is the group of all automorphisms of g and $\mathrm{Aut}_r(g)$ is the group of all semilinear automorphisms of g, then the group $\mathrm{Aut}(g)$ is normal in $\mathrm{Aut}_r(g)$ and is of index 2. An automorphism τ of g is called semilinear if $\tau(\lambda x) = \bar{\lambda}\tau(x)$ for $\lambda \in C$.

A semilinear automorphism of g is said to be the first kind if it transforms a Borel subalgebra into a Borel subalgebra of the same sign. A semilinear automorphism of g is said to be of the second kind if it transforms a Borel subalgebra into a Borel subalgebra of the opposite sign. It has been well established that any automorphism of g is either an automorphism of the first kind or automorphism of the second kind.

If g is a complex affine Kac–Moody algebra, a Kac–Moody subalgebra g_r of g is a real form of g if g is the complexification of g_r i.e. if $g = g_r + \mathrm{i}g_r$ (direct sum). Such a real form g_r determines a mapping $C\colon g \to g$, namely $x + \mathrm{i}y \to x - \mathrm{i}y (x, y \in g_r)$. The mapping C has the following properties:

(i) C is semilinear, i.e. $C(\pi x + \mu y) = \bar{\pi}C(x) + \bar{\mu}C(y)$, for $x, y \in g$ and $\pi, \mu \in C$.
(ii) C is an involution i.e. $C^2 = 1_g$
(iii) $C[x, y] = [Cx, Cy], \qquad \text{for} \quad x, y \in g.$ (2.3)

A map C: $g \to g$ with these properties is a bijection called a semilinear involution of g. Conversely any semilinear involution C of g uniquely determines a real subalgebra $g_r = \{x \in g; Cx = x\}$ such that $g = g_r + \mathrm{i}g_r$, that is to say a real form of g. The real form g_r of g is said to be almost split if it is associated with a semilinear involution of first kind and is said to be almost compact if it is associated with a semilinear involution of second kind.

Thus, we see that a semilinear automorphism of order 2 of g is called a semi-involution of g. For any semi-involution σ', we have the decomposition $\mathrm{Aut}_r(g) = \{1, \sigma'\} \otimes \mathrm{Aut}(g)$ ($\otimes$ denotes semidirect product). We denote by σ'_{n} the conjugation of g with respect to the standard split form. We call σ'_{n} the standard normal semi-involution of g. This commutes with the standard Cartan involution ω.

Let $\omega'_{\mathrm{s}} = \sigma'_{\mathrm{n}}\omega = \omega\sigma'_{\mathrm{n}}$. Then ω'_{s} is called the standard Cartan semi-involution of g. Its algebra of fixed points is the standard compact real form of g. A Cartan semi-involution of g is a semi-involution ω' conjugate to ω'_{s}, by an element of $\mathrm{Aut}_r(g)$. Then ω' is a semi-involution of the second kind and the associated real form is called compact real form t_1 of g.

Let σ' be a semi-involution and ω' be a Cartan semi-involution. If σ' and ω' stabilize the same Cartan subalgebra h, one may suppose by conjugating by G that ω' commutes with σ'.

Let σ' be a semi-involution of g of the second kind and let $g_r = g^{\sigma'}$ be the corresponding almost compact real form. A Cartan semi-involution ω', which commutes with σ' is called a Cartan semi-involution for σ' or g_r. The involution $\sigma = \sigma'\omega'$ (resp. its restriction ω'_r to g_r) is called a Cartan involution of σ' (resp. of g_r). The algebra of fixed points $t_0 = g_r^{\sigma}$ is called a maximal compact subalgebra of g_r. We have the Cartan decomposition $g_r = t_0 \oplus p_0$ and $t_1 = t_0 \oplus \mathrm{i}p_0$, where p_0 is eigenspace of ω'_r for the eigenvalue -1.

We obtain under $\mathrm{Aut}(g)$ a one–one correspondence between the conjugacy classes of (linear) involution of the second kind of g and the conjugacy classes of almost split real forms of g. Now, we consider

(1) The semi-involutions σ' of the second kind of g.
(2) The involutions θ of the first kind of g.
(3) The relation $\sigma' \approx \theta$ if and only if,
   (a) $\omega' = \theta\sigma' = \sigma'\theta$ is a Cartan semi-involution.
   (b) θ and σ' stabilize the same Cartan subalgebra h.
   (c) h is contained in a minimal σ'-stable positive parabolic subalgebra.

Then this relation induces a bijection between the conjugacy classes under $\mathrm{Aut}(g)$ of semi-involutions of the second kind and conjugacy classes of involutions of the first kind.

Thus, we obtain under $\mathrm{Aut}(g)$ a one–one correspondence between the conjugacy classes of (linear) involutions of the first kind of g (including identity) and the conjugacy classes of almost compact real forms of g. The compact real form is unique; it corresponds to identity. A classification of involutions of an affine Kac–Moody Lie algebra is given by Levstein [13]. A classification of automorphism of finite order of affine Kac–Moody Lie algebra of type $A_n^{(1)}$ is given by Kobayashi [12]. A description of automorphism of Kac–Moody algebras is given by Kac and Wang [11] and Cornwell *et al* [21, 22].

Now, let C be the semilinear involution of g defined by g_r, so that $C(x + \mathrm{i}y) = x - \mathrm{i}y$ for $x, y \in g_r$. Now C acts on the root spaces as follows. For each root $\alpha \in R$, (R being the set of all roots of the Kac–Moody algebra), define $\sigma(\alpha)$ by

$$-(\sigma(\alpha))(x) = \overline{\alpha(C(x))} \qquad \text{for} \quad x \in h, \qquad \text{where } h \text{ is a Cartan subalgebra} \tag{2.4}$$

then

$$C(g^{\alpha}) = g^{-\sigma(\alpha)}. \tag{2.5}$$

Table 1. $A_2^{(1)}$. Involutive automorphisms from Satake diagrams.

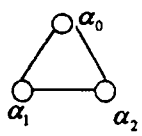
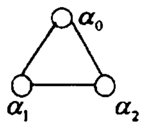
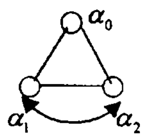
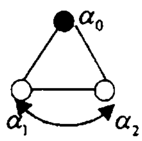
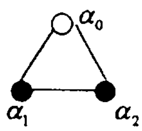
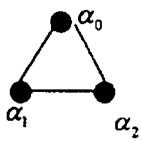
Dynkin diagram	Satake diagrams	Involutive automorphisms
		$-\sigma(\alpha_0)=\alpha_0$ $-\sigma(\alpha_1)=\alpha_1$ $-\sigma(\alpha_2)=\alpha_2$
		$-\sigma(\alpha_0)=\alpha_0$ $-\sigma(\alpha_1)=\alpha_2$ $-\sigma(\alpha_2)=\alpha_1$
		$\sigma(\alpha_0)=\alpha_0=\beta_0$ $-\sigma(\alpha_1)=\alpha_2+\alpha_0$ $-\sigma(\alpha_2)=\alpha_1+\alpha_0$
		$-\sigma(\alpha_0)=\alpha_0+2\alpha_1+2\alpha_2$ $\sigma(\alpha_1)=\alpha_1=\beta_1$ $\sigma(\alpha_2)=\alpha_2=\beta_2$
		$\sigma(\alpha_0)=\alpha_0=\beta_0$ $\sigma(\alpha_1)=\alpha_1=\beta_1$ $\sigma(\alpha_2)=\alpha_2=\beta_2$

Table 2. Restricted root systems from Satake diagrams.

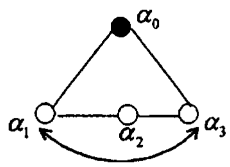
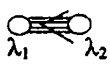
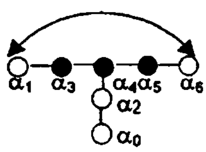
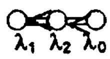
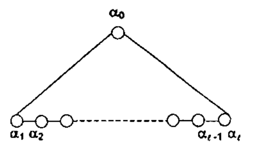
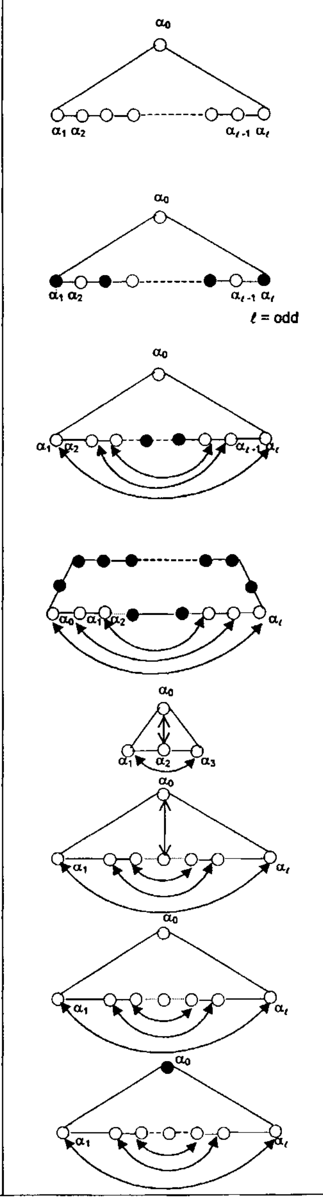
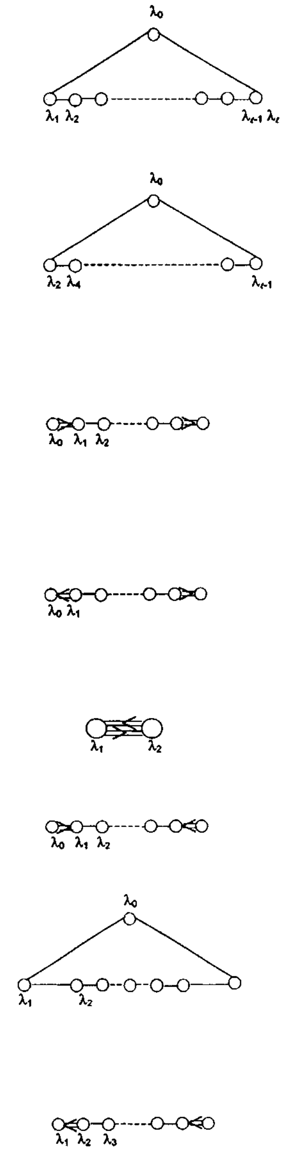
	Satake diagrams	Dynkin diagrams of Restricted Root System
$A_3^{(1)}$		
$E_6^{(1)}$		

Table 3. Affine untwisted Kac–Moody algebra.

TYPE	DYNKIN DIAGRAMS	SATAKE DIAGRAMS	DYNKIN DIAGRAMS OF THE RESTRICTED ROOT SYSTEM
$A_\ell^{(1)}$			

The mapping $\alpha \to -\sigma(\alpha)$ extends by linearity to an involutary isometry under which $R_- = R - R_0$ is stable and R_0 is the set of roots $\alpha \in R$ such that $\alpha - \sigma(\alpha) = 0$. Therefore, we are led to consider pairs (R, σ), where R is a root system and σ is an involutary isometry such that $\sigma(R) = R$. The pair (R, σ) is said to be normal if $\alpha \in R$ implies $\alpha + \sigma(\alpha) \notin R$. Each real Kac–Moody algebra determines a normal pair (R, σ) which determines g_r up to isomorphism.

Table 3. (Continued.)

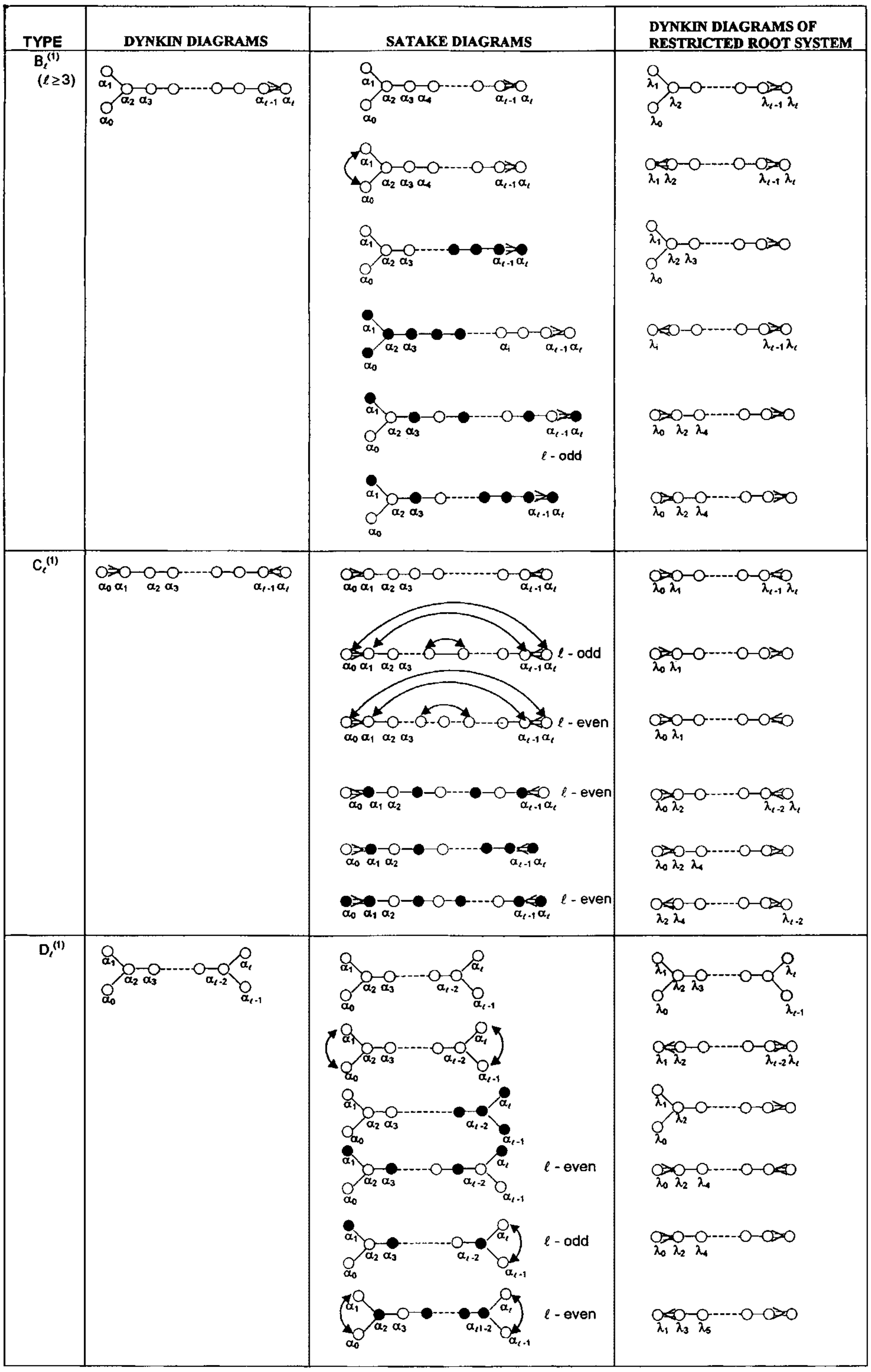
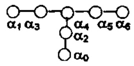
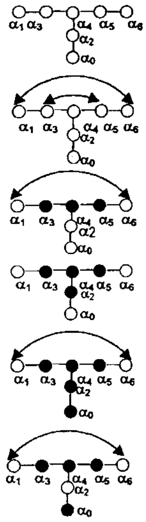
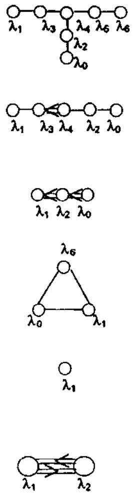
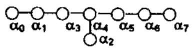
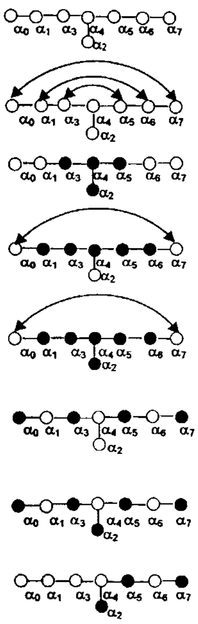
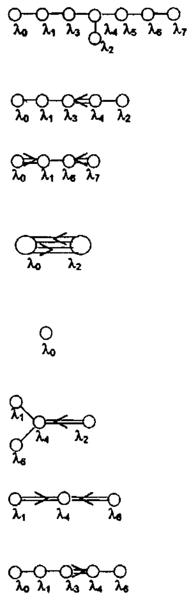
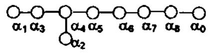
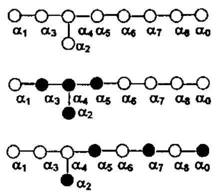
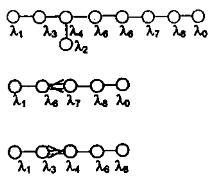
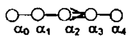
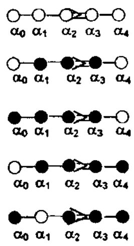
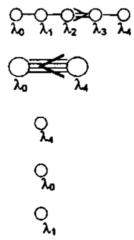
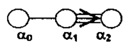
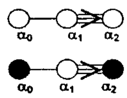
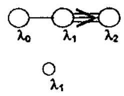


Table 3. (Continued.)

TYPE	DYNKIN DIAGRAMS	SATAKE DIAGRAMS	DYNKIN DIAGRAMS OF RESTRICTED ROOT SYSTEM
$E_6^{(1)}$			
$E_7^{(1)}$			

This condition is used in determining the Satake diagrams and the corresponding involutive automorphisms out of all possibilities. The construction of the Satake diagrams associated with a real Kac–Moody algebra and corresponding Dynkin diagrams of restricted root system proceed as follows.

Table 3. (Continued.)

TYPE	DYNKIN DIAGRAMS	SATAKE DIAGRAMS	DYNKIN DIAGRAMS OF RESTRICTED ROOT SYSTEM
$E_8^{(1)}$			
$F_4^{(1)}$			
$G_2^{(1)}$			

Let R be a root system of an affine Kac–Moody algebra. For $\alpha \in R$, let $\lambda = \bar{\alpha} = \alpha - \sigma(\alpha)$. Let us introduce $R_- = \{\alpha | \bar{\alpha} \neq 0, \alpha \in R\}$. Also let $R_0 = \{\alpha \in R | \bar{\alpha} = 0\}$. The basis B of R can now be decomposed into two disjoint subsets, B_0 and B_-, where $B_0 = B \cap R_0$ and $B_- = B - B_0$.

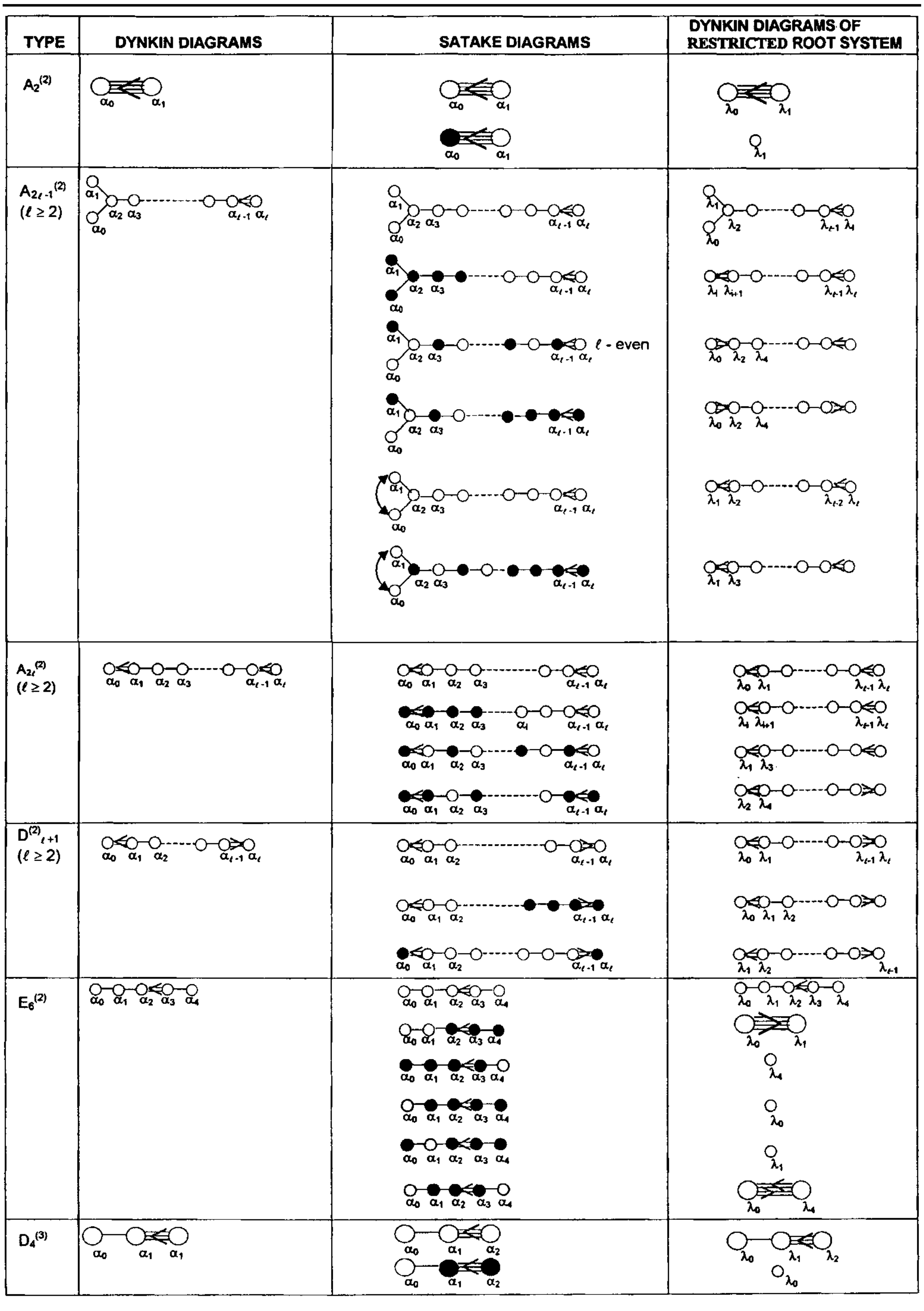
If $B_- = B - B_0 = \{\alpha_i\}$ and $B_0 = \{\beta_l\}$, then it can be shown that

$$-\sigma(\alpha_i) = \alpha_{\pi(i)} + \sum \eta_{il}\beta_l \tag{2.6}$$

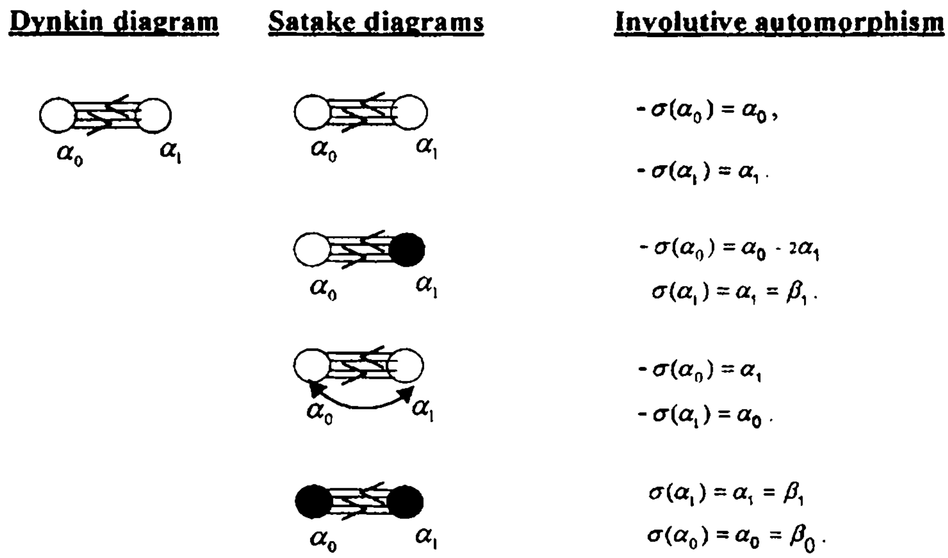
where π is an involutive permutation of $\{0, 1, 2, \ldots, r\}$ and η_{il} are non-negative integers. We can now associate with B its Satake diagrams. Denote the root α_i with white circles as usual and the root β_l by black circles. If $\pi(i) = k$, then it is indicated by a double arrow between open circles for α_i and α_k. The Satake diagram determines the involution of R uniquely. We note that $\sigma(\beta_l) = \beta_l$ and if $\alpha \in R$ then $\alpha + \sigma(\alpha) \notin R$. If all the vertices of the Satake diagrams of g are black circles, then the real form is compact. If all the vertices of the Satake diagrams of g are white circles and if the diagram contains no arrows, then the real form is split. All other real forms lie in between these two extremes. Let (R, σ) be a normal pair with R being a root system in a vector space V. Let $V = V_+ + V_-$ be the direct sum decomposition of V into eigenspaces for $-\sigma$ corresponding to eigenvalues ± 1. Then the collection $\Sigma = \{\bar{\alpha} | \alpha \in R\}$ forms a root system in V_+, which are referred to as a restricted root system. These vectors are essentially projections of the roots R onto V_+ and we can choose simple roots from Σ to form a restricted Dynkin diagram.

To illustrate our points, we have taken $A_1^{(1)}$ and $A_2^{(1)}$ as two examples and have shown how to draw the Satake diagrams of these algebras and obtain corresponding real forms.

Table 4. Affine twisted Kac–Moody aglebras.



(a) $A_1^{(1)}$: the generalized Cartan matrix for this algebra is $\left(\begin{smallmatrix}2 & -2\\ -2 & 2\end{smallmatrix}\right)$. The Dynkin diagram, the Satake diagrams and the involutive automorphisms of $A_1^{(1)}$ are as shown below.



Thus, we see that we have four real forms of this algebra each corresponding to one type of involutive automorphism, which is similar to the results obtained earlier [12, 21–23].

(b) $A_2^{(1)}$: the generalized Cartan matrix for this algebra is $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. The Dynkin diagram, the Satake diagrams and the corresponding involutive automorphisms are as given in table 1.

Thus, for $A_2^{(1)}$ we see that we have five real forms of this algebra each corresponding to one type of involutive automorphism [12, 21–23].

Once the Satake diagrams are obtained it is very easy to construct the corresponding Dynkin diagrams of the restricted root system as mentioned previously in this section. To illustrate our point, the Dynkin diagrams corresponding to the restricted root system obtained from the Satake diagrams of $A_3^{(1)}$ and $E_6^{(1)}$ are given in table 2.

In the case of ordinary Lie algebras these Dynkin diagrams correspond to symmetric spaces, which have many applications as mentioned earlier. But in the case of affine Kac–Moody algebras, we do not know what type of physical and mathematical significance these diagrams will have. However, we are trying to address these aspects in a forthcoming paper.

Now we list Satake diagrams together with Dynkin diagrams of restricted root system associated with all untwisted as well as twisted affine Kac–Moody algebras in tables 3 and 4.

3. Conclusion

In this paper, we have demonstrated how Satake diagrams are one of the easiest tools to classify the restricted root system associated with affine Kac–Moody algebras. Since symmetric spaces/real forms are related to the involutive automorphisms of Lie algebras, this technique may also provide a basis for the determination of symmetric space analogues (if they exist) and real forms associated with affine Kac–Moody algebras. The Satake diagrams themselves can be exploited to find the involutive automorphisms which can be used to determine directly the Iwasawa and Langland's [15–18] decomposition of the corresponding algebras (particularly low rank). The same technique can also be used with suitable modification to determine the real forms and symmetric super spaces associated with simple Lie (super) [19] algebras as well as that of affine Kac–Moody (super) algebras [20]. Such types of studies are currently in progress and will be communicated in future. We hope the restricted root systems determined from such type of studies may have direct physical relevance with quantum integrable systems and quantum transport problems which is beyond the scope of this paper.

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